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ON SOME THEORIES DUE TO PARETO, ZAWADSKI, W. E. JOHNSON AND OTHERS

[THESE " Recent Contributions to Mathematical Economics " (as they were entitled when published in the ECONOMIC JOURNAL, March and June 1915) are divided, not very sharply, according to a distinction suggested by some of the writers reviewed, into two parts dealing respectively with the general theory of economic equilibrium and " tentative applications of mathematical economics outside the general theory of equilibrium." To the first class belong Mr. W. E. Johnson's original representation of supply and demand free from a certain particularity which confines the usual representations; also the analogy which is drawn between Producers' and Consumers' surplus; and the treatment of utility or satisfaction as an object of economic science with the countenance of the great mathematician Poincaré. But we can hardly describe as belonging to the general theory of economic equilibrium a *curiosum* which is pointed out by Zawadski after Pareto and independently by Mr. W. E. Johnson : namely, that where the demand for commodities is correlated (in the way of rivalry) a rise in the price of one of the articles (due to a tax, for instance) may cause an *increase* in the consumption of that article. The proposition should rather be referred to the chapter of tentatives, if it can be supposed to have a bearing on practice. More important examples of applied—or applicable—mathematical economics are afforded by Professor Pigou's theory that production tends to be a maximum when the marginal productivity is equal in every branch; his proof that the elasticity of the demand for common labour is in general highly elastic—a tenet favourable to Free Trade; his doctrine of Joint Cost, showing that the " Cost of Service " principle—the rule proper to simple Competition—is appropriate in cases which had previously been supposed subject to the rule of " What the traffic will bear."]

RECENT CONTRIBUTIONS TO MATHEMATICAL ECONOMICS

Théorie Mathématique de l'Echange. ANTONIO OSORIO. Ave une introduction de Vilfredo Pareto. Traduit par Jos D'Almada. (Paris: Giard. 1913.)

Principes d'Économie Pure. La Théorie de l'Échange sous le régime de la libre concurrence. E. ANTONELLI. (Paris: Rivière. 1914.)

Les Mathématiques appliquées à l'Économie Politique. W. E. ZAWADSKI. (Paris: Rivière. 1914.)

The Vagaries of Recent Political Economy. Professor J. S. NICHOLSON. (*Quarterly Review*, October, 1913.)

The Pure Theory of Utility Curves. W. E. JOHNSON. (*Economic Journal*, December, 1913.)

Contributo alla Teoria dell' Offerta a costi congiunti. MARCO FANNO. (Rome: *Athenæum* Supplement to *Giornale degli Economisti*, October, 1914.)

The purport of these pages is to report the progress in mathematical economics which may have been made since last a contribution to the subject was noticed in the ECONOMIC JOURNAL, namely, June 1913. We do not define the limits of the subject strictly. "To clear up the relations of fundamental economic conceptions"—in the words applied by one of our authors to the work of Cournot and his successors—is no doubt the principal achievement, but we are not prepared to say that it is the only hopeful employment, of mathematical economics. We do not, with M. Antonelli, restrict the subject to what he and other of our authors describe as the general theory of economic equilibrium. We rather follow M. Zawadski, who indeed makes that general theory his main object, yet adds a chapter on "tentative applications of mathematical economics outside the general theory of equilibrium." So we in our *first* and main section will confine ourselves to the commonly recognised territory, the domain proper, of our science; but in a supplementary *second* section we shall examine the zone of influence extending beyond that territory.

SECTION I.

The subject of this section may be broken up—anatomically so to speak and for the purpose of demonstration, though not in

life and fact—into three parts. *Firstly*, we shall consider the pure theory of exchange, making abstraction of the concrete fact that most things exchanged have been produced. *Secondly*, we shall introduce the circumstance of production and observe what progress has been made in what has been called the Mechanics of Industry. Corresponding to the rôle of Energy in the theory of Mechanics is the predominance of utility or satisfaction in mathematical economics. The conception is indeed immanent in all our reasonings; but it may be artificially isolated for special consideration in a *third* sub-section.

(1) *Theory of Exchange.*—This part of the subject has received special attention from three of our authors. Their predilection and success may be traced to the influence of the economist who first stated the theory of exchange in all its generality, Léon Walras. M. Antonelli's work, indeed, may be regarded as an abridgment of Walras's *Éléments d'économie pure ;* a task which the illustrious author himself had commenced. This work of a disciple is valuable as a clear and simple exposition of the founder's doctrine. M. Osorio's treatise is based not only on the work of Walras, but also on that of his distinguished successor, Professor Pareto. The treatise is, we may be sure, a most valuable addition to the economic literature of Portugal. The translation into French has to encounter the formidable rivalry of the original writers. To what extent the free adaptations of their theories are to be considered improvements will depend partly on the concurrence of the reader with M. Osorio in attributing "extreme concision" to "Walras's and Pareto's deductions." M. Zawadski also is deeply imbued with the doctrines of those original writers. But he is not bound to adopt the words of any master. He views the school of Lausanne in just relation to other schools, of which he had taken a comprehensive survey.

In his statement of the problem M. Zawadski makes an assumption about the data which we regard as important. For the purpose of the abstract theory—which "affords an approximate image of the real phenomenon"—we should think of "dealers who often meet having each at his disposal [*porteurs de*] about the same quantity of goods" (p. 200). Again, "the economic tendencies imaged by the abstract theory are the more decided (*plus prononcées*) the more the phenomena are regular, continually repeated under analogous conditions" (p. 201). And again, with special reference to *production :* "Only transactions frequently repeated under analogous circumstances can present a certain

character of regularity, so it is only to these that our theory is in general applicable " (p. 215).[1]

M. Zawadski is also happy in his view of the function of money, or, in Walras's more general terminology, *numéraire*, in a theoretical market. He starts with the assumption that one out of m commodities plays the part of money, and so in a perfect market there are $(m - 1)$ prices; but he does not regard this proposition as axiomatic, rather as deducible (by way of " arbitrage ") from a greater number, conceivably $m(m - 1)$, rates-of-exchange between different commodities (p. 134, p. 124). We may add that even the existence of a uniform rate-of-exchange between any two commodities is perhaps not so much axiomatic as deducible from the process of competition in a perfect market (X. 85).[2]

However this may be, we incur no serious loss of generality in postulating that one out of the m commodities acts the part of *numéraire*. If there are n individuals each buying or selling any number of the m commodities, it is beautifully shown that to determine the state of equilibrium there are given as many equations as there are unknown quantities, namely, the $(m - 1)$ prices and the amounts of each commodity (including the money) acquired positively or negatively, so to speak, by each of the n individuals—mn (in addition to the said $m - 1$) quantities—on the ideal supposition of each individual dealing in all the commodities. Here, as throughout his work, M. Zawadski appears to us to present the cream of Walras's and Pareto's thoughts.

He justly claims for Walras priority with respect to the general theory of exchange. But he appears to us to do less than justice to the *doyen* of English economists when he suggests that the interdependence of economic quantities is not recognised in the *Principles of Economics* (Zawadski, p. 307). Surely Dr. Marshall has adequately presented this great truth in passages relating to " joint demand " and " composite demand," to " joint supply " and " composite supply "; for instance, in Note XXI. of his Mathematical Appendix. There Dr. Marshall affirms that cardinal principle which is the main outcome of Walras's teaching :

[1] A cognate conception has thus been expressed (see **B**, I., p. 40) in relation to the labour-market : " On the first day a set of hirings is made which proves not to be in accordance with the dispositions of the parties. These contracts terminating with the day, the parties encounter each other the following day, with dispositions the same as on the first day—like combatants *armis animisque refecti*—in all respects as they were at the beginning of the first encounter, except that they have obtained by experience the knowledge that the system of bargains entered into on the first occasion does not fit the real dispositions of the parties."

[2] The references of this form relate to the list of writings given below at p. 477.

"However complex the problem may become, we can see that it is theoretically determinate, because the number of unknowns is exactly equal to the number of equations which we attain." In the text to which that note refers and in many other passages [1] Dr. Marshall reaffirms the interdependence in question. He is quite aware that a demand-curve representing the connection between two economic quantities—in particular price and the amount demanded—may become less serviceable than usual—may, for instance, ascend with a rise of price—when the commodity demanded is related in a certain peculiar and exceptional way to some other commodity.[2] But he is also aware that "there are very few practical problems in which the corrections to be made under this head would be of any practical importance." He does not think it necessary to repeat continually that "the neglected elements would generally belong to the second order of quantities." In short, he appears to have assigned to the doctrine just the amount of space which is due to it in a treatise not primarily concerned with mathematical abstractions.

The sin of omission is with more plausibility imputed to professedly mathematical writers. In M. Zawadski's otherwise too flattering notice of some articles dealing with Monopoly which have appeared in this Journal there is made—by implication and cross-reference—an objection apparently identical with that to which we have alluded. The "superiority of Pareto's solution" (p. 203) over Cournot's is affirmed on grounds apparently common to Cournot's *Recherches* and the Articles in question. With respect to Cournot's equation for the determination of price in a regime of monopoly [3] it is objected: "It is not this equation by itself which determines the price of the commodity by itself, but it is this equation simultaneously with all the others pertaining to the system which determines all the prices and all the quantities exchanged" (p. 303). "The demand ['débit'] for a commodity is a function not simply of its price, but also of the quantities bought and the prices of the other products. These latter cannot be considered constant, theoretically at least, when the corresponding magnitudes relative to the first commodity are varied" (p. 58, note; compare p. 60, note). The dilemma stated with respect to the use of plane curves appears, therefore, to be applicable here. Either "the other economic quantities

[1] See, in particular, *Principles of Economics*, 6th edition, pp. 100, 105, 130, 132.

[2] See below, pp. 460, 479.

[3] $\frac{d[p_y f(p_y)]}{dp_y} = 0$, in M. Zawadski's notation.

exercise no influence on the quantities offered or demanded [of the quantity under consideration] "; or " the other quantities are constant " (p. 299). We are not concerned to defend Cournot—" Deorum injuriæ Diis curæ "; but we demur to the description as applicable to the articles in the ECONOMIC JOURNAL. In the first of them (IV. 56, 234) the transactions contemplated are thus typified : " Suppose three islands, A, B, C, engaged in this sort of international trade. A imports from B goods for the manufacture of which B has to import materials from C." The consumers in A, forming a monopoly, dictate the price of the goods which, regard being had to the quantity of imports forthcoming at each price, affords the greatest advantage to the monopolist purchaser. It is not assumed that the other economic magnitudes, such as the price and quantity of the materials imported from C, have no influence on the price of the product exported from B, nor that they remain constant.[1] On the contrary, attention is directed to the influence of the other quantities. In short, the dilemma is escaped by the process which M. Zawadski describes as the " third eventuality " (*loc. cit.*, note); the relation between the price and the offer is determined by the elimination of the " other quantities." The example is no doubt a particular case. The monopolist is a *buyer ;* and the total quantity of the factors of production is for a purpose in hand supposed constant. Complications of demand and supply are expressly abstracted ; " correlation of supply or demand not being now supposed " (V. 234). That complication is, however, mentioned in the immediate context ; and elsewhere in the series of papers referred to has been frequently, in the view of some inordinately, recognised. Altogether the example of the principle on which monopoly price is determined may be taken as fairly representative.

We do not expect that our author will press his objections to the " third eventuality "—the elimination of the economic quantities other than the one which is immediately under the control of the economist—when it is explained that the process is virtually equivalent to " Pareto's Solution " (Zawadski, p. 202, last par. ; p. 220, last par.) ; abridged, it may be, for the purposes of practice or exposition. We do not suppose that there is any substantial difference between our author's position and our own. We quite agree with him in holding that when complicated

[1] See (VI.) referred to in the ECONOMIC JOURNAL (V.). The monopolist purchaser there considered (p. 19) " will go on varying p_1 " [the price of the article purchased corresponding to M. Zawadski's p_x] directly, and indirectly p_2 [the price of the factor of production, one of M. Zawadski's " other economic quantities "] by means of an equation which might be used for eliminating p_2.

eliminations of " other economic quantities " are necessary, " the process would leave us in the most complete ignorance of even the elementary properties [such as, we presume, the *descending* character of the demand curve] of the resulting functions " (*loc. cit.*, p. 300). We notice with complete approval his tacit postulate that something more than theoretical determinateness, " certain additional properties of the functions," as he elsewhere says (p. 165), may reasonably be looked for. In this connection he has some very wise observations, which we have not space to transcribe, on the possibilities and limits of mathematical economics (*loc. cit.* and p. 187). We gather that it is not the only, though a principal, use of this study to show that there exists a determinate solution of the problems, the equations being neither more nor less in number than the unknowns. Even with reference to the " general theory of equilibrium," even in the limited and recognised sphere to which our first section is restricted, some additional propositions may be expected, some laws of contract (I. 146) which might be available in practice if we only had corresponding minor premises. Such hypothetical applications are presented by M. Zawadski in his section on the *laws of individual demand and offer* (pp. 180–187), and by Mr. Johnson in his article on *Utility-curves*. We shall endeavour to present some salient features of these theories, without attempting to make an abstract of writings which hardly admit of compression—one of them being adequately, and the other exceedingly concise. Comments and corollaries, more than copies, will be offered.

We may begin by recalling the construction which Dr. Marshall employed many years ago to illustrate the theory of international trade.[1] Let the amount of goods imported by a country (such as the " cloth " of Mill's example) be X, and let the equivalent amount of exports (such as " linen ") be Y. A point (X, Y) represents an international exchange; the inclination (to one of the axes) of the line joining the point to the origin represents the rate of exchange between exports and imports. The Demand-and-Supply Curve pertaining to a country is the locus of points at which trade may be in equilibrium. Subsequent writers have applied the construction to exchange in general; and in particular to the case now under consideration, exchange without reference to production. It has been shown that the

[1] In the papers put under contribution by Pantaleoni in his *Principii di Economica Pura*. They may be seen, as we are informed by the learned M. Zawadski, in the Goldsmiths' Library at the University of London. The construction is reproduced and discussed in the ECONOMIC JOURNAL (IX. (b). *Cp.* VIII. 70). See below, p. 477.

Demand-and-Supply Curve may be regarded as the locus of points at which a straight line passing through the origin touches an *indifference-curve ;* that curve being the locus of points representing bargains between which there is nothing to choose. To fix the ideas we might suppose the indifference-curves to consist of concentric circles with centre C in Fig. 1. The person to whom the construction relates—conceivably a typical individual—desires always (or at least for all the cases which our construction is apt to represent, say the space bounded by the axes through O and, on the right, by a perpendicular through C) to pass from any point at which he may be placed to another on an indifference-curve nearer the centre. But he has no interest in the change

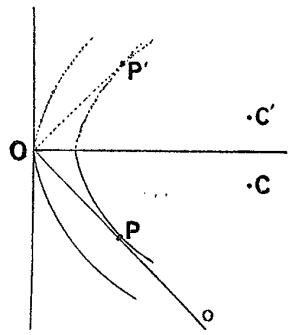

FIG. 1.

of terms represented by movement from one point to another on the same indifference-curve. The curves thus defined are also and will here usually be called "utility-curves."

Now let X be an addition to an initial quantity x_o, and likewise Y be subtracted from y_o; x_o and y_o being the co-ordinates of O relatively to an origin below and left of O, say the left corner at the bottom of the page. Let x (measured from that origin) $= x_o + X$; $y = y_o - Y$. Then in order to represent in terms of x and y the series of bargains which have been above described, it is proper to substitute for any point at a distance Y above the horizontal through O a point at the same distance *below* that horizontal (the horizontal distance from O, viz. X, remaining unchanged). Thus the utility-curves in the transferred construction will consist of concentric circles about a centre C' (which is *above* the horizontal through O).

The convexity (towards the axes of x and y) of the utility-curves which are thus presented is not accidental; it is an important essential property. It follows from the axiomatic, or at least commonly experienced, circumstance that an individual possessing x_o of one commodity and y_o of another will in general find it his interest to give some of one commodity in exchange for some of the other; and that if there is fixed rate of exchange between the commodities there will be a point x, y at which it will just cease to be to his interest to exchange at this rate. Accordingly at that point, as shown in the figure, the utility-curve must be convex to the line $O\ o$ representing the rate of exchange —the " price-line " in Mr. Johnson's phrase—and accordingly, as the line descends from left to right, convex to the axes of x and y, say, as before, the bottom and the left edge of the page. In the figure the transaction represented is the giving of some of commodity (y) in exchange for some of commodity (x).[1] But if the initial amounts are represented by some point on the same price-line *below* O, say o, p would still be that point at which the individual would cease to do business at the assigned rate.

The analytic condition that the utility-curve should be convex brings into view a nice distinction, of which Mr. Johnson makes much use. In discussing his theories we would gladly imitate the parsimony of symbols practised by another of our authors. We should like to draw the line at *partial differential coefficients*.[2] But unfortunately the reasoning turns mainly on the conceptions connoted by those coefficients. To avoid the use of the appropriate symbols would be open to the criticism which Todhunter has passed on that section of Laplace's introduction to his Theory of Probabilities in which, for the benefit of the general reader, the operations of the higher mathematics are expressed in ordinary language. Todhunter describes the section as " a complete waste of space." " It would not be intelligible to a reader unless he were able to master the mathematical theory delivered in its appropriate symbolical language, and in that case the section would be entirely superfluous." [3] We can only practise temperance, not abstinence, in the matter of symbols.

Let $f(x, y)$ ($\equiv u$) = constant be the equation to a utility curve. The slope of the tangent at any point x, y on the curve is

[1] The *genus*, as distinguished from *a quantity* of a commodity, is expressed by the use of brackets.

[2] More exactly partial *finite differences*—differences not always very small, absolutely or relatively to the variable (e.g., x) to which the difference (Δx) relates (XI. 565).

[3] *History of Probabilities*, p. 497.

$-\left(\frac{df}{dx}\right) / \left(\frac{df}{dy}\right)$[1]; where the bracketed differential coefficients are *partial*. The condition that the curve should be convex (to the axes) at the point (x, y) is that the (complete) differential coefficient of this expression for the slope, which we may call t, should be positive. This condition may be written $a\theta_1 + b\theta_2 < o$,[2] where a and b are positive quantities, θ_1 and θ_2 are made up of partial differential coefficients derived from the utility-function, u. In order that the inequation should be satisfied, one or other of two alternatives must hold good; we must have either (α) both θ_1 and θ_2 negative, or (β) one of them only negative, while the other is positive (the limiting case in which one of them is zero being neglected). The first alternative (α) is properly described by Mr. Johnson as the "standard case"; the second (β), the exceptional alternative, is shown by him to have some important properties.

His theory relates principally to the case of three variables, two commodities and money. But we may introduce the subject without considering more than two variables. Let x be the quantity of a certain kind of commodity that is purchased at the price ξ. Let z be the purchase-money taken from an initial store of money μ; so that after the purchase our man has $\mu - z$, say Z, money to expend in other ways. Now if the utility of commodities other than (x) is entirely independent of the quantity of (x) consumed, we may consider the total utility realised, say u, as the sum of two terms, say $f(x) + F(Z)$. This total is to be maximised *subject to the conditions that* $z + Z = \mu$ (a constant). In other words, there is to be maximised $u - m[x\xi + Z - \mu]$; where m is the undetermined coefficient proper to problems of *relative* maximum, the expression within the square brackets being equated to zero. We have, then, to determine the three quantities x, Z, and m, the three equations

$$(1)\ \frac{df}{dx} = m\xi;\ (2)\ \frac{dF}{dZ} = m;\ (3)\ x\xi + Z = \mu.$$

[1] Brackets outside partial differential coefficients will be sometimes omitted where no doubt can arise. A sloping line is used to denote *division*.

[2] $\theta_1 \equiv \left(\frac{du}{dy}\right)\left(\frac{d^2u}{dx^2}\right) - \left(\frac{du}{dx}\right)\left(\frac{d^2u}{dxdy}\right)$; $\theta_2 \equiv \left(\frac{du}{dx}\right)\left(\frac{d^2u}{dy^2}\right) - \left(\frac{du}{dy}\right)\left(\frac{d^2u}{dxdy}\right)$;

$a \equiv 1 \Big/ \left(\frac{du}{dy}\right)^2$; $b \equiv \left(\frac{du}{dx}\right) \Big/ \left(\frac{du}{dy}\right)^3$. From the essential properties of a utility function it is evident that a and b are both positive.

It may be noted that in Zawadski's (Pareto's) notation (p. 171), $\phi_y\phi_{xx} - \phi_x\phi_{xy} \equiv \theta_1$; $\phi_x\phi_{yy} - \phi_y\phi_{yx} \equiv \theta_2$. Mr. Johnson's $\frac{dV}{dx}$, $\frac{dW}{dz}$ differ only by positive factors respectively from our θ_1 and θ_2.

Accordingly, if the data are changed, in particular if the price ξ is raised, we are in a position to determine how the variables x and Z will be affected thereby. If we consider only a small change of ξ we may obtain (differentiating each of the equations by ξ) three equations for the three differential coefficients $\frac{dx}{d\xi}$, $\frac{dZ}{d\xi}$ and $\frac{dm}{d\xi}$, whence it is deducible that with the increase of ξ x must decrease, z may increase or decrease.

This familiar proposition is introduced here only as a stepping-stone to the less simple case in which the expenditure *on* x and *of* Z are not independent of each other, some of the commodities on which Z is to be expended being correlated [1] to, and in particular competitive with x. On this supposition it will be proper to put for u, $f(x, Z)$ [2]; and to then proceed as in the simple lemma. We have now three equations for the aforesaid three differential coefficients with respect to ξ.[3]

Eliminating two of the variables, we obtain for $\frac{dx}{d\xi}$, the *increase* of x corresponding to an increase of its price, an expression of the form $-a + b\theta_2$; where a and b are positive quantities (not the same as those lately employed) and θ_2 is the quantity above defined (Z now being substituted for the commodity y), which is generally negative, but occasionally positive. In the latter case the quantity purchased, x, *may increase* with the rise of price. In that case it is evident from equation (3) that Z must decrease.

This reasoning may easily be turned so as to treat explicitly of two commodities; supposing that the whole available income [4] is to be expended on two [5] commodities. We have only to suppose that Z now represents the amount of a second commodity of which

[1] The term "correlated" being used for the genus comprising complementary and competitive. See Index, s.v. *Correlation*.

[2] "f" as here used to denote a function involving *money* as one of its variables is not quite on a par with the "f" in the context, which denotes a function of commodities used—a utility function. The former kind of function is to be conceived as obtainable from the latter kind (by proper eliminations).

[3] (1) $\left(\frac{d^2f}{dx^2}\right)\frac{dx}{d\xi} + \frac{d^2f}{dxdZ}\frac{dZ}{d\xi} - \xi\frac{dm}{d\xi} = m;$

(2) $\frac{d^2f}{dxdZ}\frac{dx}{d\xi} + \frac{d^2f}{dZ^2}\frac{dZ}{d\xi} - \frac{dm}{d\xi} = x;$

(3) $\xi\frac{dx}{d\xi} + \frac{dZ}{d\xi} = -x.$

[4] As here used, μ denotes the whole available income (not a part arbitrarily earmarked to expenditure on two commodities); otherwise the change in the marginal utility of money as measured by $\frac{du}{d\mu}$ would not be significant.

[5] More concretely two kinds of commodities.

the price is *unity* (as may be supposed without serious loss of generality).

The reasoning may be extended to three variables if we put $u = F(x, y) + f(Z)$ where x and y are the amounts of two commodities which are purchased for the sum of money z at the prices ξ and η, and $Z + z =$ constant, the maximum of u being thus relative to the condition $x\xi + y\eta + Z =$ constant. We leave it to the reader to deduce conclusions similar and additional to those which we have enounced with respect to two dimensions. For the general analysis, which we have adapted to some simplified cases, the reader is referred to Professor Pareto's *Manuel*, or to M. Zawadski's abridged, but lucid, exposition (p. 180 *et seq.*).

Some interesting results *in pari materiâ* are obtained by Mr. Johnson with the aid of analytic geometry. To introduce his theories let us extend to three dimensions the construction in two dimensions above described. Let us now suppose a system of utility-*surfaces*; such as concentric spheres with centre in the positive quadrant bounded by the planes xy, xz, yz. The construction is suited to represent two commodities of kinds (x) and (y) purchased in quantities x and y for the sum of money z; it being understood that z is a variable conditioned by equations like those written in the last paragraph. If the prices of (x) and (y), ξ and η respectively, are given, the amounts demanded and offered, the value of the variables x, y, z are determined. The system of values x, y, z is represented by the point at which one of the utility-spheres is touched by the *price-plane*, as we may say. This is a plane parallel to one passing through the origin of x and y and through two lines in the planes (xz), (yz) respectively corresponding to the given prices, the lines of which the equations are $z = x\xi$, $z = y\eta$. As in the case of two dimensions the system of quantities x, y, and ξ (quantity of commodity purchased, purchase money and price) becomes determinate when any *one* of them is given, or more generally any one equation connecting them; so in the case of three dimensions the system of quantities x, y, z, ξ, η becomes determinate when any *two* of them, or any two equations connecting them, are given. As in the case of two dimensions when not one variable or equation is given, we can construct a curve, the demand-and-supply curve, connecting one of the variables with another or with a function of the two others (x with y, or with y/x—the rate of exchange $\equiv$ say $\tan\theta$); [1] so in the case of three dimensions when one and only one variable, or equation

[1] Or, using polar co-ordinates, we may connect θ with ρ ($\equiv \sqrt{x^2 + y^2}$).

connecting the variables, is given we can construct a curve in three dimensions representing the course of the three quantities x, y, z as one of the other quantities, in particular one of the prices, is varied. The projection of that curve on the plane (xy) exhibits the change in the magnitudes of x and y consequent upon a change of one of the data.

Mr. Johnson has employed that plane curve to contemplate the variation of x and y in two or three instances which lend themselves particularly well to analytical geometry. Thus, suppose the one equation given to be this: that the ratio between the prices, ξ and η, is constant. For example, we may suppose a general rise or fall of prices not materially altering the relation between the prices of the different commodities; while incomes remain constant. To determine how a certain individual (or a particular class of individuals) will respond to such a change, we may write as the equation of the demand-and-supply curve in the plane of (x, y) in our notation, $t =$ constant.[1] Accordingly, for the slope of this curve we have $-\left(\frac{dt}{dx}\right) \Big/ \left(\frac{dt}{dy}\right)$. Therefore in general (α) the slope is positive, the values of x and y both decrease when the prices rise (both increase when the prices fall). But in the exceptional case (β) when one of the θs is positive more of one commodity and less of another will be demanded.

The construction may also be applied to the following problem. Suppose that the amount of money at the disposal of the individual, the total which we have called μ, is increased (or diminished), the utility-function remaining in other respects unaltered. What will be the effect on the quantities purchased, x and y; the prices ξ and η now both remaining constant? As before, it may be shown that in general both purchases will be augmented; but in the exceptional case when one of the θs, *e.g.*, θ_1 (in our notation), is positive, the case of "x urgent" in Mr. Johnson's terminology, much more of x but less of y will be purchased.

"What is the value of such a conclusion?" asks M. Zawadski (p. 186), with reference to his own demonstration that a fall of price may be attended with a diminution in the amount purchased. "Does the case really occur, otherwise, than by way of exception?" We can only answer, with Dr. Marshall,[2] "Such cases are rare; when they are met with each must be treated on its own merits." A good preparation for that requisite treatment

[1] Above, p. 459.

[2] *Principles of Economics*, Book III. ch. vi. § 4, p. 132, 6th edition.

appears to be afforded by the exercises in abstract reasoning provided by Professor Pareto and Mr. Johnson. We shall return to the subject under our third head. Under the present head we shall only add some observations on certain leading economic conceptions which Mr. Johnson has employed successfully.

In the construction for the representation of exchange between two commodities the plan of measuring the amount of one given in exchange *downwards* [1] (negatively) has certain advantages. It enables the indifference- or utility-curves from which the demand-curves are derivable to be expressed more simply; utility being treated as a function of the quantities consumed. The utility-curve does not now vary with variations in the conditions of the market, such as the amounts of each commodity in the hands of the dealer at the opening of the market. There is avoided a certain unreal heterogeneity imparted to the indifference-curves when the amount given is measured positively. When thus represented, the curves necessarily appear to belong to two distinct types, whereas conceivably the law of utility might be identical for buyers and sellers. In order that exchange should take place it is only necessary that the distribution at the opening of the market should be different. Thus in Fig. 1, if the point O represented the position of half the market, say $\frac{1}{2}n$ dealers, and the point o the position of the other $\frac{1}{2}n$ dealers, all having identical *laws* of utility, there would be an exchange characterised by the point P, supposing that $OP = Po$. If the initial distribution were different there would still be (in general) exchange; though it might happen that some who had been buyers under the first conditions would become sellers under the second.[2]

This praise is not to be understood as disparagement of the older constructions for the representation of exchange between the two kinds of commodity. No doubt the last-mentioned advantage on the side of the newer representation is considerable, at least when we leave international trade proper, to which Dr. Marshall's curves were originally adapted, and apply them to exchange in general. The advantage, however, may be partly secured by supposing two groups between which there is a certain mobility so that the numbers on each side of the market may vary according to circumstances.[3] The intersection of the two demand-and-supply curves pertaining respectively to buyers and sellers is particularly well calculated to bring out the principal outcome of mathematical

[1] Above, p. 457.

[2] The construction proposed by Mr. Wicksteed in the ECONOMIC JOURNAL fo September, 1913, seems designed to secure this sort of advantage.

[3] *Cp.* X. 88.

theory, the determinateness of economic equilibrium. The constructions mentioned in the last paragraph are not so well suited for this purpose. The case there put, in which half the market would be massed at one point, is quite imaginary. Usually the initial positions of the dealers would correspond to points scattered over the plane. The geometrical representation of this system would have little advantage over algebraic analysis—such as that which is given by M. Zawadski, after Walras and Professor Pareto—when reduced so as to apply to two commodities.

Mr. Johnson does not claim superiority for his construction as a representation of exchange between *two* commodities. It is the case of *three* articles, one of them being money, which he has specially illuminated. His work appears to us to be in respect of mathematical technique a unique contribution to the subject. In a remarkable degree geometrical elegance is coincident with economic importance.

Mr. Johnson has thrown additional light on the peculiar case which we have called β;[1] a case to which Professor Pareto had already called attention.[2] Mr. Johnson proposes to define "competitive" as distinguished from "complementary" commodities by the characteristic of class β; from which it follows that "in the former case changes along the demand-curves [of the kind described above] involve an opposite variation in x and z [our y], in the latter the two increase or decrease together" (Johnson, p. 496). This subtle distinction is now to be compared with the simpler definition which was proposed some years ago (V., 21), and appears to be accepted by M. Zawadski (p. 173). According to this definition, two commodities of the kinds (x) and (y) are, for the quantities x and y, complementary or competitive according as the partial differential coefficient[3] which represents the increase in the marginal utility of one commodity consequent on the gratuitous acquisition of an increment (or small unit) of the other commodity is positive or negative; in symbols, according as $\frac{d^2u}{dxdy} > \text{or} < o$. It will be observed that the class *complementary* as thus defined is entirely included within the larger class which Mr. Johnson denotes by that term. For when $\frac{d^2u}{dxdy}$ is positive both θ_1 and θ_2 are negative.

[1] Above, p. 459.

[2] *Manuel*, p. 573. *Cp.* Zawadski, p. 170.

[3] See note 2 to p. 458. It often depends on the magnitude of the differences (Δx, Δy) under consideration whether a case is to be classed as Joint Production or not (XIII. 566).

Mr. Johnson's class *competitive* also includes a part of our class *competitive;* the greater part, indeed, since β is admitted to be an exceptional phenomenon. Which of these two definitions is preferable?

In favour of the older definition it may be urged that a distinction so fundamental should not be made to depend upon the incidents of a comparatively advanced regime, such as the distribution of money among different purchases, or even the more elementary distribution of doses of labour among different kinds of production. The distinction should be more *intrinsic;* one which Robinson Crusoe might have drawn between articles found by him on the derelict ship. Among such, for example, were barrels of powder, "fowling-pieces," and pistols. The first two commodities were complementary in our sense, since by the acquisition of the one the other became more useful. But the relation between guns and pistols was different; for the more of one article that was acquired, the less the other would be a desideratum. Might not these relations be usefully predicated, although the Johnsonian criterion would not be available? Or, again, consider the rice and barley to the cultivation of which Robinson Crusoe directed his primitive agricultural labours. Might he not decide whether one food went well with the other or the reverse, without waiting to observe whether, in case he obtained command over a greater quantity of labour (say through the accession of Friday), he would require (for his own consumption) more of both commodities, or much more of one and less of another?

On the other hand, as most of us live under a monetary regime, a definition relating thereto may well be useful. Propositions relating to money form a large part of economic theory. But of that part a large proportion does not admit of the distinction on which Mr. Johnson dwells. For the possibility, or at least the significance, of class β presupposes variability in the marginal utility of money.[1] But very generally, with reference to internal trade at least, the marginal utility of money may be treated as constant.[2] It is, therefore, not surprising that many of the deductions which have been made respecting commodities correlated in the way of demand presuppose constancy in the value of money (VI. 21) and employ only the older definition (V., *cp.*

[1] If m is constant, it is clear that when μ (the outlay in money) is increased (prices being constant) all that will happen is that more both of x and of y will be purchased.

[2] *Cp.* Marshall, *cited* above, p. 39, note. M. Zawadski, indeed, complains that constancy in the marginal value of money is too freely postulated by economists.

note 2, p. 468, below). In international trade, indeed, the value of money is normally variable. Probably with respect to some difficult problems in international values, the new conceptions, if not the new terminology, may be particularly appropriate.

We shall have more to say about this matter under the head of Production. We may also postpone some further points relating to the exchange of consumable commodities until we have considered Production—a subject closely connected in Mr. Johnson's treatment with Consumption.

Theory of Production.—The formulæ which have been obtained in relation to consumption may be transferred almost unaltered to production. As the consumer seeks to lay out any assigned sum of money so as to obtain a maximum of satisfaction, so the producer seeks to incur a minimum of cost in producing any assigned quantity of product. The mathematical expression of the conditions is almost identical; at least with reference to a single product, in the absence of joint production. Whether is it easier to say that

$$f(x, y) - m[x\xi + y\eta - \mu]$$

shall be a maximum, where μ is an assigned outlay of money and the other symbols have the same signification as before; or to say

$$(x\xi + y\eta) - \frac{1}{m'}[f(x, y) - v]$$ [1]

shall be a minimum; where now x and y are quantities of factors-of-production—*two* doing duty as representative of any number—ξ and η are their prices, v is the quantity of a product connected by a "production-function" (in Mr. Johnson's phrase) with x and y, $1/m'$, like m, is the constant proper to a relative maximum (or minimum).[2]

The theorems which have been proved for utility-functions

[1] The form of the function f is, in general, to be determined by the Calculus of Variations. Or, what comes to much the same, we may consider v as depending not only on the factors of production, x, y, etc., but also on certain adjustments defined by quantities which may be called "gratuitous constants" (XIII. 357); for instance, the *time* or *place* at which some operation is to be performed—within limits not affecting the total cost—might well be constants of the character. Say $v = \phi(x, y; p, q)$ to each of the gratuitous constants $p, q, \ldots$ there corresponds an equation $\frac{dv}{dp} = 0$; by means of which equations the said constants may be eliminated. [The "gratuitous constants" in the production-function are cognate to the coefficients or "parameters," u, v, w, which Dr. Zotoff introduces and eliminates in his interesting development of Mr. Johnson's theory (ECONOMIC JOURNAL, Vol. XXXIII. (1923), p. 115.]

[2] More exactly the quantity to be maximised is subjective, the utility obtained by the entrepreneur from profits less by the disutility attending production (cp. II., note h).

may be transferred by analogy to production-functions. In particular the mathematical distinction between *complementary* and *competitive* goods should be the same, whether the goods are, in the terminology of the Austrian School, of the first or of a higher order. Here, then, we have a new argument against the suggested innovation in the definition of these terms. For, with reference to production, it is equally tenable that the terms should designate a difference which is *intrinsic*. We need not now go to Robinson Crusoe's island for an illustration. Whenever an industry is *integrated*, in the sense that the factors of production and the product are manufactured by the same firm, it may be important to distinguish between the cases in which the increase of one factor increases the efficiency of another factor, or the reverse. But that distinction is expressed by the sign of the partial differential coefficient $\frac{d^2v}{dxdy}$ rather than by the signs of expressions corresponding to the quantities which we have called θ_1 and θ_2.

The constant $m'\left(\frac{1}{\kappa}\text{ in Mr. Johnson's notation}\right)$ is well described by him as the marginal efficiency of money. In fact, the increment of the product effected by applying a (small) unit of money to increase any factor (e.g., x) is measured by the price of the factor multiplied by m'; since $\left(\frac{dv}{dx}\right) = m'\xi$. For any assigned quantity of product v there are, when the prices $\xi, \eta \cdots$ are given, determinate values of $x, y \ldots$, and accordingly a determinate value of μ, the total expenditure, $= x\xi + y\eta \cdots$. We have thus $\mu = \chi(v)$ where v is the product corresponding to Mr. Johnson's "p" (p. 507); and χ is a function corresponding to the "cost-curve" defined elsewhere [XIII. 362, and context]. For the price of the commodity we may put the reciprocal of the marginal efficiency of money $1/m'$; abstracting not only the circumstance that there is an interval of time between the outlay on the factors and the completion of the production, but also the distinction between prime and general expenses and other circumstances of high importance on any but the most abstract view of the subject. On this hypothesis the exchange value of the outlay (μ) in terms of the commodity produced (v) would be $\mu\frac{dv}{d\mu}$. The excess of the product over the cost thus measured would be $v - \mu\frac{dv}{d\mu}$.

In applied mathematics (as Mr. Johnson reminds us, p. 503)

we are often concerned not so much with the simple differential coefficient, the relation of two increments, say $dv : d\mu$, as with the ratio between the *relative* changes $dv/v : d\mu/\mu$; in short, with the *elasticity* $\frac{\mu}{v}\frac{d\mu}{dv}$. The last written elasticity has a very important property. According as this elasticity, say ϵ, is greater or less than unity, the ratio v/μ increases or diminishes as μ increases. But the increase or decrease of that ratio forms the criterion of *increasing returns* [1] in a common and very important sense of the term. If ϵ is less than unity industry cannot be in a state of equilibrium, in a regime of competition.[2]

It is pointed out by Mr. Johnson that, as production is increased with the increase of outlay, the " marginal efficiency " of money changes, but does not necessarily increase or diminish (p. 510). In this connection he investigates the condition that the production-function should be such as to afford a true minimum (of cost). He shows that as production—and therewith outlay—is increased the locus (in space of many dimensions generally) of the system of simultaneous values $x, y \ldots$ (in our notation) is " analogous to a line for cutting across the equipotential surfaces " $v = f(x, y \ldots)$ (p. 509). That family of surfaces are shown to be in all directions *convex* to the axes of (our) $x, y \ldots$. This last statement may seem inconsistent with the illustration of progressively increasing production which was given in the former paper, to which reference has been made. There the cost attending the use of the factors was represented by a *plane* (XIII. 365). It must be remembered, however, that the analysis there offered—much less rigid than Mr. Johnson's—refers specially to *monopoly*. In a regime of monopoly equilibrium may very well be reached, though the production-function has not the normal convexity. It should be added that the analysis in the context referred to is specially directed to the explanation of a particular incident, *increasing returns*. Mr. Johnson's more general theory is adapted to wider applications.

The analogy between consumption and production, between maximising utility and minimising cost, is calculated to elucidate

[1] See XIII. 354 *et seq.* (Below, p. 477.)

[2] See XIII. p. 358; and *cp.* Johnson, p. 507. The difference between our statements and Mr. Johnson's as to the elasticity ϵ is explained by the fact that we, in accordance with the very abstract suppositions which may be ascribed to the Continental writers, suppose an entrepreneur producing such a relatively small part of the aggregate output as not to affect (through the action of increasing or diminishing returns) the price of the product. Mr. Johnson's formulæ are doubtless appropriate to the practically more important case of *long-period supply-curves* (*cp.* XIV. 6, par. 1, and note 1; VIII. 66, and context).

one or other of the phenomena, whichever is the less clear and familiar. It is usually the more subjective of two compared phenomena which gains in clearness by the comparison. Yet the poets, those masters of allegory, occasionally illustrate things of sense by things of soul. A skylark is "like an unbodied joy." The way of Phæacian ships was "like a thought." So those to whom the working of their own minds are more familiar than the ways of business men may be helped by the proposed analogy to understand the nature of entrepreneurs' profits. They may be encouraged to question the paradox propounded by the school of Lausanne and repeated by M. Zawadski that "the entrepreneur *quâ* [en tant qu'] entrepreneur makes neither gain nor loss." Those who would uphold this tenet are, in virtue of the said analogy, placed under the heavy burden of having to prove that the consumer *quâ* consumer obtains no pleasure. The consumer's surplus of utility, of the form $F(x, y) - m\mu$, is by common consent not equatable to zero. Why should we equate to zero the producer's surplus of product (the total product *minus* the equivalent in product of the amount of money laid out), viz. $f(x, y) - m'\mu$? [1] We know no more about the function F than about the function f.[2] Competition, it may be said, presses upon profits. But so it does upon utility; the net advantages in different occupations being reduced to a level by industrial competition (in Cairnes' phrase). That level may be low; the remuneration of the average occupied person, measured in the pleasure that money can command, may in fact be small. But that it is normally *zero* neither common sense nor economic theory compels us to believe.

So far abstracting (among other concrete circumstances) the *general expenses* of a business; which it is interesting to note that Walras left out of account. If they are taken into account, the argument becomes *a fortiori*. For why should not a

[1] That is, upon the very abstract suppositions above specified (p. 468, note). With respect to most of the circumstances abstracted it may be observed that their great importance was first pointed out by Dr. Marshall. The type of industry formed by their abstraction is identical with the conception entertained in a paper to which M. Zawadski alludes (p. 205. *Cp.* Osorio, p. 28), a paper (II) written before the appearance of the *Principles of Economics*. The deficiency there noticed (p. 688, and *note* h) in the then prevailing mathematical systems in relation to *Industrial* (as distinguished by Cairns from *Commercial*) competition has since then been remedied by Marshall's "*Long-period*" *Supply-curve* and Pigou's *Curve of marginal supply prices*.

[2] It was perhaps the exigency of the theory in question which led a distinguished economist to maintain that the product was a *homogeneous function* of the factors of production (VII. 182), and has led other theorists to make by implication statements about the function which are only less preposterous because less distinct.

substantial remuneration for the entrepreneur be included in the general expenses of the business? In fact, that is probably the meaning of the more moderate disciples, if not the leaders,[1] of the School of Lausanne. If the paradox is only a figure of speech, ' solvuntur risu tabulæ." The entrepreneur is transformed, like the father of the Bourgeois Gentilhomme, who, it was discovered, had after all not been a shopkeeper. 'Tis true he was a very good-natured and obliging man, and as he was a connoisseur in drapery he used to get together goods of that sort and make presents of them to his friends—" pour de l'argent."

M. Zawadski writes (p. 211): " L'entrepreneur réel apporte dans son entreprise des facultés et connaissances qui le distinguent des autres, son travail, son crédit, ses relations, etc.; l'entrepreneur idéal n'apporte rien de tout cela: il est absolument égal à tous les points de vue à ses collègues ou, plus exactement, pour tout ce qui le distingue, il n'est pas entrepreneur, mais fournisseur de services; il n'est que la personification de son entreprise." We do not suppose that there is any material difference between the meaning of these statements properly interpreted and our view of the matter. We regard the formulæ which are piously repeated by M. Zawadski as we do Mill's dictum that " demand for commodities is not demand for labour," [2] or any other of the paradoxical dogmas consecrated by the usage of the older English economists who, as Mr. and Mrs. Webb have remarked,[3] had " almost a genius for publishing what they did not mean to say." We, too, conceive an ideal entrepreneur who makes nothing by way of monopoly, or rent-of-ability, or " konjunctur " —though our formula is well adapted to take account of those concrete circumstances when they are present. " We may suppose that the entrepreneur's remuneration is totally unmixed with rent, so that it is open to any worker to transform himself into an entrepreneur, the difference of remuneration [between the profits of an entrepreneur and the wages of common labour] compensating for the efforts and sacrifices attending the transformation." [4]

[1] Walras by not admitting general expenses has cut himself off from this explanation; and, as remarked on a former occasion (X. 92), the theory in question appears to be for Pareto more than a *façon de parler*.

[2] Of this dictum Dr. Marshall has said " it expresses his meaning badly " (*Principles of Economics*, note on the doctrine of wages, 4th edition), and Sidgwick has said " This proposition which has occasioned a good deal of polemical discussion is, I believe, perfectly true when properly explained. . . . I think, however, it is all in form unsatisfactory. . . . I think most reflective readers of Mill find it puzzling after all the pains that he has taken to make it clear " (Sidgwick, *Political Economy*, Book I. chap. v., note).

[3] *Industrial Democracy*, Part III. chap. i.

[4] XI. 570. Other passages referring to the question are, VI. 82, VIII. 530, IX. 92.

There is thus a "supply-price," in Dr. Marshall's terminology, for the services of an entrepreneur, just as there is a supply-price of a workman's service; differing only in that the former is paid out of a surplus, the latter is commonly a marginal outlay. It is not quite clear to us how M. Zawadski would deal with the surplus which must arise [1] upon the supposition, which he at least entertains as sometimes appropriate, that the price of the product is equal to its marginal cost,[2] in the absence of general expenses (p. 212).

In this connection we should mention the coefficients of production formulated by Walras and adopted by his successors. Certainly it seems a natural conception, and agreeable to the habits of the business man, to split up the price or the cost-of-production of a (unit of) product into a number of elements each formed by the price of a factor multiplied on the amount of that factor which goes to a unit of product. Thus, in our notation, if in the state of equilibrium μ is the total expenditure, x, y, etc., are the amounts of the factors employed, n is the number of units produced, $\frac{\mu}{n} = \frac{x}{n}\xi + \frac{y}{n}\eta + \cdots$. But when the coefficients $\frac{x}{n}$, $\frac{y}{n}$ are to seek, in case of what is called "variability of the coefficients-of-production," it seems to us more natural to determine the quantities x, y, etc., with the aid of the production function $f(x, y \ldots)$, in terms of the product v, as above explained; without bringing in the coefficients of production. However, the same heights of contemplation may be scaled on different sides. In the selection of the route habit properly counts for much.

The conception which we recommend has the advantage of being readily applicable to the case of two or more products, Joint Production; with which may be coupled what has been called Rival or Disjunctive Production (XIII. 558). As before, we have to minimise the cost, $x\xi + y\eta + \cdots (\equiv \mu)$; subject now to the condition that two (or more) products, say v and w, are produced; that is, the *propositum* which is to be minimised is

$$\mu - \frac{1}{m_1}[f_1(x, y..) - v] - \frac{1}{m_2}[f_2(x, y..) - w]$$

where f_1 denotes the quantity of v and f_2 that of w which results from the application of the (amounts of the) factors x, y.. in the best available manner; [3] $1/m_1$ and $1/m_2$ are relativity-constants

[1] In the absence of unwarrantable and unworkable assumptions as to the form of the production-function (above, p. 466, note).

[2] Walras's conception as to which, see III.

[3] *Cp.* XIII. 357.

of the kind already employed.[1] Thus the prices of the factors, namely, ξ, η... being given, we have the cost in terms of the amounts of the products, say $\mu = \phi(v, w)$.

Analogous to the distinction between complementary and competitive *factors-of-production*, a distinction between *products* is now presented. Products are complementary or competitive (joint [2] or rival) according as the increase of one product alleviates or aggravates the expense of increasing the other product: in symbols, according as

$$\frac{d^2\phi(v, w)}{dvdw} < \text{or} > 0.$$

It can hardly be doubted that the above is the true distinction appropriate to difficult problems relating to railways or, more generally, "public works" (XIII. 217). Analogy and the economy of language make against Mr. Johnson's usage of the terms with respect to joint demand and its opposite.

(3) *Utility.*—Now let us turn back from the "production-function" to the analogous expression for utility, and see what new light is thrown on the more subjective conception. With reference to the difficulty of measuring satisfaction, M. Antonelli aptly quotes a letter written by the eminent mathematician Poincaré to Walras: [3]

"Can satisfaction be measured? I may say that one satisfaction is greater than another, because I prefer one to the other; but I cannot say that one is two or three times greater than another. . . . Satisfaction then is a magnitude, but not a measurable magnitude. Now is a magnitude that is not measurable therefore not amenable to mathematical theory ['par cela seul exclué de toute spéculation mathématique']? By no means. Temperature, for instance (at any rate before the term 'absolute temperature' had acquired a signification with the rise of Thermodynamics), was a non-measurable magnitude. It was arbitrarily defined and measured by the expansion of mercury. It might

[1] It may be well to remind the reader that the "best available" use of the factors does not depend on the selling-prices of v and w. *Any* two values of v and w having been *assigned* the minimising of the *propositum* affords equations enough to determine the corresponding values of x, y..., together with m_1 and m_2 (the equation to zero of each of the expressions within the square brackets being, of course, taken into the account). Of course, in order to *determine* the quantities v and w the selling-prices of those articles have to be taken into account.

[2] The subject will be treated at length in the next section, with reference to the views of Professor J. S. Nicholson and Professor M. Fanno.

[3] At the end of Walras's study entitled "Économie et Mécanique"; quoted by Antonelli at his p. 66.

quite as legitimately have been defined by the expansion of any other substance and measured by any function of that expansion, *provided that it was a continually increasing function.* Likewise, in the present case, you may define satisfaction by an arbitrary function, provided that the function continually increases along with the satisfaction which it represents." . . .

Poincaré's ruling is in accordance with the view now generally prevalent among mathematicians, that the capacity of numbers to express the results of counting and measuring "may be regarded as a secondary property derived from the more fundamental one of expressing order. Natural numbers form a series with a definite order, and the expressions 'greater than' and 'less than' mean 'more advanced' and 'less advanced' in this order." These are the words of another eminent mathematician, Professor Love.[1]

Professor Pareto is therefore in very good company when, scrupling to designate utility as a function (say u) of quantities of commodities (say $x, y..$), he contemplates a family of successive *indifference-curves* (or generally surfaces in space of many dimensions) in the plane x, y (or corresponding hyper-surface); such that the advance from any one indifference-locus to the next in succession affords an *index*, rather than a measure, of the advance in satisfaction, or as Professor Pareto prefers to say, *ophelimity*. According to M. Osorio (p. 312), not only should the combinations which are preferred have a higher index; but also if in passing from Combination I. to Combination II. one experiences a greater difference in pleasure than in passing from Combination II. to Combination III., the difference between the Indices I. and II. ought to be greater than that between the Indices II. and III.[2] The form of doctrine adopted by Professor Pareto would imply a substantial difference from received theories if the negation that u is a *function* of x, y (in our notation) in the ordinary sense of the terms involved the corollary that the system of values $x, y..$ does not normally correspond to the same amount of utility; that amount varying with the *path* by which we have attained the point $x, y..$ (starting from any initial point).[3] But it would be difficult to reconcile this possibility with

[1] See the articles of Professor A. E. Love and Professor A. Voigt referred to (IX. *a* 222).

[2] If it is objected that this statement implies the *measurability* of satisfaction, it may be replied that there are those to whom this implication does not appear a *reductio ad absurdum* (I. 60).

[3] *Cp.* Zawadski, p. 150, note (referring to Pareto, *Manuel*, pp. 547–557), p. 176, p. 209, note.

that character of repetition under similar circumstances which we have attributed to the phenomena under consideration; we do not understand that Professor Pareto would press his suggestion; and M. Zawadski, with his usual good sense, seems not to attach to it much practical significance.[1]

The matter is well put by Mr. Johnson with reference to two commodities [x and y]: "There are no lines in the figure which measure the utility itself. The several utility-curves are arranged in a scale of increasing value as we pass to the right and above [in the plane of x, y]; and thus the 'distance' (measured arbitrarily) from one curve to another 'indicates' without measuring the increase in utility. But this impossibility of measurement does not affect any economic problem" (p. 490).

Walras appears to be fully justified in the use of terms such as maximum satisfaction by the authorisation which he received from Poincaré. We have not caught the distinction on the ground of which M. Antonelli classes Walras with Cournot as dealing with objective phenomena, rather than with the "mathematico-psychological" school initiated by Gossen and developed by Pareto [2] (Antonelli, p. 17).

Identity of expression in a matter so speculative is not to be expected. But there appears to be a substantial agreement among experts that things go on as if the satisfaction obtained by an individual from an assigned set of goods was a quantity dependent on the quantities of the goods. To proceed as if there was such a dependent variable appears to be legitimate. (See Antonelli, p. 68, referring to Walras's brochure, *Économique et Mécanique;* and p. 111, "nous pourrons le supposer"; and *cp.* Zawadski, p. 154.)

If utility, say u, is a function of goods purchased for use, it follows that, prices of the goods being assigned, u is a function of μ, an amount of money which is to be expended on the purchase of those goods.[3] The differential coefficient of u with respect to μ is the marginal utility of money, which we have called m $\left(\text{Mr. Johnson's } \frac{1}{\kappa}\right)$. The relation between a *relative* (indefinitely small) change in the amount of money and the corresponding *relative* change in utility, that is $\frac{du/u}{d\mu/\mu}$ or $\frac{\mu}{u}\frac{du}{d\mu}$ is defined by

[1] *Loc. cit.*
"ce cas semble avoir assez peu d'importance . . ."
"des cas de ce genre ne jouent pas un grand rôle en pratique."

[2] The appreciative reviewer of M. Antonelli's work in the *Journal of the Statistical Society* for July, 1914, appears sensible of the obscurity which we notice.

[3] The function which is the inverse of (the analogue of) Mr. Johnson's χ (p. 507).

Mr. Johnson (p. 504) as the elasticity of u in terms of money. This coefficient plays a part in economic theory analogous to that which we have assigned to the *elasticity of money*.[1] As long as the elasticity of utility is greater than unity the ratio u/μ increases with the increase of money. Thus if the "net advantages" of an occupation (including profits) increase with the increase of investment in that line, there will be a crowding into that occupation up to a point at which u/μ, having ceased to increase, becomes equal for all occupations between which there exists "Industrial Competition" (as defined by Cairnes). The ratio u/μ, which each individual tries to maximise, is not to be confounded with the marginal utility of money. The *reciprocals* of these quantities are likewise to be distinguished; Mr. Johnson's κ, the reciprocal of our m, is not to be equated to π, which he calls the price (of a unit) of utility.

The postulate here adopted that utility or welfare "can be brought under the category of greater and less" [2] rests primarily on the testimony of consciousness, the psychological observation that there are degrees of felt satisfaction. This personal experience is then extended by sympathy to the evaluation of other people's pleasures. Jevons's suggestion that the theory of utility is limited to the motions of a single mind, that "no common denominator of feeling seems to be possible" appears to us untenable. The contrary is postulated throughout large tracts of economic science; for instance, the theory of taxation and that of industrial conciliation. Even a more fundamental part of political economy, the theory of value and distribution, involving the equation of net advantages in different occupations, suggests at least, if it does not require the comparison between, the welfare of different persons.[3] This kind of comparison no doubt presupposes some homogeneity between the persons compared, such that presumably exists between "a thousand persons living in Sheffield and another thousand in Leeds, each with about £100 a year." [4] So when we use a change in the level of prices as an index of a variation in welfare, the indications obtained are then most useful when the persons affected by the alteration of prices are of one and the same type, for instance workpeople having similar family budgets.[5]

In the example last given the rough estimation of welfare

[1] Above, p. 468. [2] Pigou, *Wealth and Welfare*, Part I. chap. i. sect. 1.

[3] See VII. 22.

[4] Part of a passage which is quoted more fully in the ECONOMIC JOURNAL (III. p. 66), from Marshall's *Principles of Economics* (3rd edition).

[5] The British Association Committee for measuring the value of money recommend construction of different index numbers for different classes.

is commonly improved by a semi-objective measure, a stable average (of percentage variations in price).

But a rough estimate would still be possible, even though the *sporadic* character proper to a good average were wanting, if, for instance, price-variations under treatment consist of two large groups, one clustering about a percentage above 100, the other about a percentage below 100. Something of the sort occurs when the price of a large item in the family budget, of house-accommodation for instance, rises while the remainder as a whole falls. A rough estimate of the change in the value of money may still be possible.[1]

This conception is sufficiently definite to enter into significant propositions. Thus it is recorded of a local dearth that, the price of bread rising very high, the price of meat and other articles fell off owing to the fact that the purchasers of those articles had to expend so much of their money on bread. In this instance, presumably, the less necessary articles followed the law of "short periods" ("market value"); the dealers sold their goods below cost price.[2] Otherwise we might suppose the prices of articles other than bread (including that of meat) to be kept constant. Under these conditions, as shown above,[3] it is conceivable that more bread might be purchased. But this occurrence is attended with a rise in the marginal utility of money.[4] In other more important cases the direction of the change in the marginal utility of money cannot be similarly predicted. In the normal conditions above designated case a, when a rise of price in one article—while the prices of other articles remain constant—is attended with a fall in the demand for the article which has become more expensive, the marginal utility of money may or may not rise. Again, suppose the income available for the purchase of different kinds of goods to be increased (as in the case adduced above, p. 402), the marginal utility of money will normally fall, but in exceptional cases may rise. In this and other respects the marginal utility of money in the way of consumption is analogous to its "marginal efficiency" in the way of production.

[1] Compare Bowley, *National Progress in Wealth and Trade*, pp. 26–7.

[2] In some previous enunciations of cognate theories (IV. and V.) a supply curve of the sort pertaining to reciprocal demand or international trade is implied (by reference to Auspitz and Lieben's constructions) so that the result of assigned changes in the price—or quantity—of one article x is given in terms of the change in the *price* of the article y as in this particular passage of the present article; not in terms of the change in *demand* for y as elsewhere in the present article.

[3] Above, p. 460.

[4] *Cp.* Marshall, *loc. cit.*

∴ Publications relating to mathematical economics, by the writer of the present article, referred to in the course of the article :—

I.—*Mathematical Psychics* (1881).

II.—*On the Application of Mathematics to Political Economy* (Report of the British Association for the Advancement of Science, 1889).

III.—" La théorie mathématique de l'offre et de la demande " (*Revue d'Économie Politique*, 1891).

IV.—" The Pure Theory of Taxation " (ECONOMIC JOURNAL, 1897).

V.—" Teoria Pura del Monopolio " (*Giornale degli Economisti*, 1897).

VI.—(*a*) Review of Bastable's *Theory of International Trade*, second edition. (*b*) Review of the same, third edition. (*c*) " Disputed Points in the Theory of International Trade " (ECONOMIC JOURNAL : (*a*) 1897, p. 397 ; (*b*) 1900, p. 389 ; (*c*) 1901, p. 582).

VII.—" Theory of Distribution " (*Quarterly Journal of Economics*, 1904).

VIII.—Review of Cunynghame's " Geometrical Political Economy " (ECONOMIC JOURNAL, 1905, p. 62).

IX.—" Appreciations of Mathematical Theories " (ECONOMIC JOURNAL : (*a*) 1907, (*b*) 1908).

X.—" On the Use of the Differential Calculus in Economics " (*Scientia*, Vol. VII., 1910).

XI.—Article on " Probability " in the *Encyclopædia Britannica* (eleventh edition).

XII.—" Applications of Probabilities to Economics " (ECONOMIC JOURNAL, 1910).

XIII.—" Contributions to the Theory of Railway Rates," Parts I. and II. (ECONOMIC JOURNAL, 1911).

XIV.—(*a*) Review of Pigou's " Wealth and Welfare " (ECONOMIC JOURNAL, March, 1913). (*b*) " Contributions to the Theory of Railway Rates : Digression on Professor Pigou's Theories " (ECONOMIC JOURNAL, June, 1913).

Section II

The theories which we have been contemplating would be demarcated by some writers from the more tentative problems to which we now proceed. Thus M. Antonelli, appreciating the work of Walras, sharply distinguishes his pure theory of economic equilibrium from his application of mathematical reasoning to bimetallism. We are more impressed by the similarity than by the difference between the more and the less general propositions which admit of mathematical treatment. Alike they are " sicklied o'er with the pale cast " of abstract thought " and lose the name of action." This similarity is indeed likely to be forgotten when, instead of x and y, some concrete matter is the subject of our theorising. When Walras prescribes for Indian currency he forgets the limitations of mathematical theory. It is not surprising that the English Government do not set much store on his bimetallic scheme, as Professor Pareto observes in the vigorous preface which he contributes to M. Osorio's volume. We heartily agree with Professor Pareto when, in this connection, he says : " Anyone who expects [" *veut* "] to derive the solution of a practical problem simply and solely [" *exclusivement* "] from the theories of pure economics, or even those of applied economics, is generally wrong " [" *est généralement dans le faux* "] (*loc. cit.*, p. xvii). But while agreeing that both the pure mathematical theory and that which has the semblance of being applied are nearly equally false in a certain sense, if taken too literally, we also think that they may be nearly equally useful as showing probability or tendency. As M. Zawadski says of the pure theory, which predicates maximum " ophelimity " of free competition : " It does not follow that [because it cannot be applied directly] it is altogether without bearing on practice [" *soit privée de toute portée pratique* "] (p. 289). As a champion of Free Trade Professor Pareto has assuredly derived support from the principle of maximum satisfaction, as it is called in English; though he is aware that the unqualified assertion of *laisser faire* is folly (Zawadski, p. 288, note). We submit that a similar use may be made of less general propositions. There is attained a presumption analogous to that scientific common sense, that almost unconscious record of experience, which underlies many of the theorems of Probabilities.[1] The designation " *a priori* " or " unverified " which has been applied to such presumptions is

[1] See, on p. 477, *ante*, XIV. 225; XIII. 205; XII. 287, 459, 463, and earlier writings there cited.

not intended to cut from under them the ground of experience, but rather to mark the absence of that third stage in the "Concrete Deductive Method" which Mill called "Verification."[1] The presumptions which we postulate are analogous to the "antecedent probability" which Mill (after Laplace) employs in the calculation of chances—"it would be impossible to estimate that probability with anything like numerical precision," yet "we may be able to form a conjecture" adequate to sustain "a practical conclusion."[2] Such presumptions are of the kind which "would naturally be assumed" in Dr. Marshall's words "to start with, . . . and until cause to the contrary were shown."[3]

Of this kind is the presumption that the imposition of a tax will diminish the demand for a commodity. It is almost sufficient to say that the negative would be violently contrary to common sense[4]; calculated to excite the derision which was bestowed by an economist of the highest sagacity upon the cognate paradox that a tax upon a (monopolised) article might prove beneficial to the consumer.[5] Common sense in the example before us is further justified in the case of a *small* tax by a presumption resting on a higher, more expert, sort of common sense, that the marginal utility of money may be treated as constant. When the tax is so large as to render this presumption hazardous, we must fall back on the first presumption, strengthened by a consideration of the conditions on which the rise or fall of price depends. It will be seen from the formula which expresses those conditions (above, p. 460) that the consumption of the taxed article will be diminished when its correlation in the way of demand with untaxed articles is small. Now this is a datum, like Mill's "antecedent probabilities," about which we may be able, in his words, to "form a conjecture" sufficiently accurate for "a practical conclusion." Some other conclusions of less practical importance may be gathered from the study of the formula.

It is difficult to formulate the presumptions of common sense

[1] *Logic*, Book III. ch. xi.; Book VI. ch. ix.

[2] *Logic*, Book III. ch. xviii. § 6.

[3] *Principles of Economics*, Book III. ch. vi. § 3, p. 30; referring to the assumption that "a shilling's worth of gratification to one Englishman must be taken as an equivalent with a shilling's worth to another." The passage including the note should be studied in its bearing on the treatment of utility as a quantity (*ante*, p. 58 *et seq.*).

[4] Thus Professor Carver expresses a generally and almost universally valid belief when in his scholarly paper on "The Shifting of Taxes" (*Yale Review*, November, 1896), he says: "It is scarcely conceivable that a tax can increase the demand for the thing taxed." Yet we have seen that this hardly conceivable, is not impossible.

[5] XII. 296 *et seq.*; and earlier writings there cited.

so unequivocally as not to admit of being misrepresented and misapplied by captious critics and stupid practitioners. There must ever be understood a saving clause like that which Aristotle appended to his definition of moral virtue, "ὡς ἂν ὁ φρονιμός." The nature of the presumptions postulated may best be exhibited by examples. We shall take these from Professor Pigou's *Wealth and Welfare*, a work which abounds in "tentatives" (as they would be described by M. Zawadski), going beyond the received applications of mathematical method.

The same work has been utilised by the writer of the article in the *Quarterly Review*, which we have cited, to point a different moral. But the difference is perhaps not so great as it appears. The reviewer, indeed, maintains that "between this pure science and the application of practice there seems to be a deep gulf fixed." Whereas, we attempt to bridge that gulf. But we are careful to put up a conspicuous notice to the effect that "this bridge is not adapted to carry heavy traffic." Our "antecedent probabilities" will generally require to be strengthened by concrete materials in order to lead to practical conclusions. It is not clear that the polemic in the *Quarterly Review* is intended to demolish the inchoate construction which we describe. The attack is rather directed against the form of exposition adopted by a particular writer. The questions involved pertain mostly to literary criticism. How far is it advisable to employ a technical terminology? What is the happy mean between abruptness and diffuseness of style? There is room for some diversity of judgment on such questions. But we do not venture to oppose our judgment to that of one who is a master of the arts of exposition. We are not, however, precluded from citing in illustration of a particular point passages which have been criticised on grounds not relevant to that particular point.

Our first example consists of the leading principle which Professor Pigou thus enunciates in the form of two propositions. "The first is that the dividend necessarily stands at the maximum attainable amount when the marginal net product of resources is equal in all uses; the second, that self-interest, if not interfered with, tends to make these marginal net products equal." [1] Professor Pigou may be open to criticism for having alluded to Adam Smith's corresponding theory as "highly optimistic," without mentioning the serious qualifications of the general theory which were introduced by Adam Smith (*Quarterly Review*, p. 420). We do not venture to dispute about Adam Smith's meaning with the

[1] Pigou, *Wealth and Welfare*, Part II. ch. iii. § 1.

editor of the *Wealth of Nations*. We are ready to admit that *Wealth and Welfare* might be improved by fuller references; and we may add, by a more complete index. But we are not here concerned with the form of the treatise. What interests us is the substantial identity between the leading principles enunciated by Professor Pigou, and those which propounded by Adam Smith have revolutionised the world of industry and commerce. What the *Quarterly Reviewer* disparagingly describes as "Professor Pigou's translation of the plain language of Adam Smith into the language of marginal net products" appears to us an improved restatement of fundamental doctrines—a revised version which, though not comparable in respect of style with the authorised version, has the advantage in respect of accuracy. The mathematical statement brings more clearly into view the essential characteristic of a maximum. What the critic suggests (*loc. cit.*, par. 2), and what others more loudly proclaim, that the mathematical statement of a general proposition involves a neglect of practical limitations, is not, we submit, true of Professor Pigou.[1] In the immediate context of the leading principle which we have cited he introduces exceptions, and throughout makes it clear that he treats the general propositions of political economy as "truths only in the rough," as Mill says. Many of the exceptions which he points out are of that *recherché* species which the mathematical method is peculiarly adapted to discover, thereby "making clear how far we are from being able to solve with full knowledge of the case a multitude of questions which are boldly decided every day." These words of Cournot are quoted with approbation by the writer in the *Quarterly Review*, in an earlier writing,[2] where he shows how the mathematical economist, "by making clear the nature and extent of the assumptions implied in dealing with

[1] If it be objected that in the above cited enunciation of leading principles Professor Pigou uses the word "necessary" and in several other passages an italicised "*must*," it may be replied that Adam Smith, who admittedly recognises practical limitations, is yet very fond of the word "necessary." For instance, in connection with the principle now under consideration:

"The study of his own advantage naturally, or rather necessarily, leads him to prefer that employment which is most advantageous to the Society" (Book V. ch. ii. par. 4).

"Every individual who employs his capital in the support of domestic industry necessarily endeavours so to direct that industry that its produce may be of the greatest possible value" (*loc. cit.*, par. 7).

"The industry of the country therefore is thus turned away from a more to a less advantageous employment, and the exchangeable value of its actual produce instead of being increased, according to the valuation of the law-giver, must necessarily be diminished by every such regulation" (par. 12).

[2] Transactions of the Faculty of Actuaries, Vol. IV. Part I. *The Use of Mathematical and Legal Ideas in Economic Problems*. By J. Shield Nicholson.

economic problems . . . invites the statesman to proceed with caution."

It often happens, as we have had occasion to notice, that a mathematical writer who enounces some *recherché* exception is in consequence supposed to be denying the general rule.[1] It is hard that he should be also suspected of affirming the general rule unreservedly as if it did not admit of exceptions. It was said of some narrow-minded specialist that he was a man who could see a fly upon a barn door without being able to see the door or the barn. A double degree of blindness seems to be attributed to the mathematical economist; now represented as incapable of seeing the barn, and now the fly. The truth appears to be that the relation between the large and the small, the general and the particular,[2] is better conceived by one who has been trained in mathematics, including probabilities, than by one whose soul this science has not taught to stray beyond generalisations of the Ricardian type. However this may be, it is certain that whoever employs general propositions in economics, whether expressed mathematically or not, is exposed to the suspicion of neglecting facts; especially on his first appearance, and before he may have acquired a reputation for caution and good sense. Adam Smith himself is no exception to this rule. Adam Smith stands accused of "Smithianismus." Nor is it only to pedants of the German historical school that Adam Smith has appeared too abstract, but also to practical English genius. It was Burke who said,[3] "You, Dr. Smith, from your professor's chair, may send forth theories upon freedom of commerce as if you were lecturing on pure mathematics; but legislators must proceed by slow degrees." Fox was not so much impressed as the *Quarterly Reviewer* is by the limitations with which the generalisations of Adam Smith are guarded. For, as reported by Lord Colchester, "in talking of books upon political economy he said (as I have often heard him say in debate) that he had but little faith in Adam Smith, or any of them, their reasons were so plausible but so inconclusive."[4]

Imputations of this sort have, no doubt, sometimes been

[1] *Cp.* F., p. 144.

[2] The relation is well expressed in the following passage, one of a large class of similar conclusions: "All that we have proved is that situations are *possible* in which a diminution in the falsity of judgment or a diminution in the costs of movement will make marginal net products more unequal. When, however, we are contemplating, from a general point of view, the consequences of these diminutions it is not the *possible* but the probable effect which concerns us" (*Wealth and Welfare*, Part II. ch. iv. § 10).

[3] As related by (the second) Lord Lansdowne, *Hansard*, 1820, Vol. I. p. 550.

[4] *Diary of Lord Colchester*, Vol. II. p. 7.

deserved, not, indeed, by Adam Smith, but by some of his followers, for instance, Miss Martineau. But our withers are unwrung. We have been careful to explain with Professor Pareto that the solution of practical problems is not to be expected from economic theory pure and simple. We do, indeed, claim that general propositions of the kind which Adam Smith qualifies with the adverb "necessarily," may afford general directions which are useful "to start with," and in the absence of knowledge to the contrary. We should not expect the first principle now under consideration, whether as stated by Adam Smith or in Professor Pigou's version, to be of much avail in an emergency, say for the direction of a committee providing employment for the wives of absent soldiers. But with reference to some wider question *in eodem genere*, to one taking a general view of women's work, the principle may well be significant. Suppose it to be ascertained that, as Mr. Sidney Webb finds probable, "women's work is usually less highly paid than work of equivalent difficulty and productivity done by men." [1] *Pro tanto*, the productive resources of the community would not be distributed so that marginal net products should be equal. Against such a distribution there is the presumption that in the words of Mrs. Sidney Webb,[2] "it is by the fullest possible use of all the productive faculties of the whole population that we shall obtain the largest yield of services and commodities." This presumption is of great importance. It is of the kind which—tempered with common sense and regard to fact—has worked mighty revolutions in industry. Yet the presumption is not by itself decisive. Before giving play to the Smith-Pigou principle, we require, having regard to the concrete circumstances, marriage and domestic life, to be secured against the danger of that process of degradation through subsidised competition which is described by Mrs. Webb as "industrial parasitism." [3]

As a second example of an economic tendency let us take Professor Pigou's proposition that the elasticity of the aggregate demand for labour is much greater than unity.[4] There is room for difference of opinion as to the form in which this presumption

[1] ECONOMIC JOURNAL, Vol. I. p. 635 *et seq.* Compare Mrs. Fawcett, ECONOMIC JOURNAL, Vol. II. p. 174; arguing that women are crowded into classes of industry less remunerative than those open to men.

[2] *The New Statesman*, July 25, 1914; one of a series of valuable articles on "Personal Rights and the Woman Movement."

[3] Cp. *loc. cit.*, Aug. 1.

[4] *Wealth and Welfare*, Part II. ch. ii. § 11 *et passim*. The proposition is criticised in the *Quarterly Review*, *loc. cit.* p. 417, and defended by Professor Pigou in the next number of that Review (Jan. 1914).

should be stated. So experts differ about the statement even of geometrical axioms. But we submit that some such general proposition, resting on the sort of evidence to which we all along appeal, underlies the received arguments in favour of labour-saving machinery and free trade. How else can the free trader reply to specious objections like those employed by Byles in his *Sophisms of Free Trade*.[1] The objections urged in former numbers of the ECONOMIC JOURNAL against some of the arguments in favour of free trade employed by Professor Bastable in his *International Trade* are similarly to be answered by presumptions as to the way in which productive forces probably act. It was objected that the transaction between the employing and employed class in a country is of the same genus as international trade, that a removal of barriers to the trade between nations may well—and not infrequently does—prove permanently injurious to a particular nation; and therefore that the removal of restrictions on importation into a particular country may well prove permanently detrimental to the employed class as a whole. The answer is to be sought in common sense, and probabilities founded on general experience. An answer in this sense has been given by the objector himself.[2]

[1] *E.g.*, "Suppose stockings to the value of £500,000 a year are made in Leicester and exchanged annually for gloves to the amount of £500,000 a year made in Dover. . . . Suppose now the Leicester people instead of exchanging their stockings for gloves from Dover exchange them for gloves . . . say from Calais. Dover loses what Calais gets. . . ." *Sophisms of Free Trade*, Edition 1904, p. 26.

[2] "In economics it is often difficult to hold fast general resemblances without ignoring—or appearing to ignore—specific differences. In the present matter, while apprehending that the transactions between the operative and the employing classes are of the genus international trade, we must not forget that the exports and imports of this trade are of a very peculiar character. The peculiarity might be partially illustrated by the trade which used to flourish between England and the Southern States of America; these States exported to England raw cotton, receiving in return cotton manufactures. If the offer of raw cotton with the demand for cotton manufactures were to be increased on one side of the international market by a change such as the growth of population in the Southern States, other things being the same, the offer of manufactures on the other side of the market on the part of a large and flourishing England would be likely to keep pace with the offer of raw material, in such wise as not to alter the terms of international exchange to the disadvantage of the average Southerner. But indeed, the illustration hardly does justice to the expansiveness of the trade which we are now considering. Let us rather suppose the export to consist of that rawest and most extensively demanded material, mechanical power. Let us imagine, for the sake of illustration, Niagara harnessed in the service of man to belong wholly to the United States, not in part to Canada; and that by improved means of transmitting force the means of production may be conveyed from Niagara to any department of Canadian industry. If the supply of power from Niagara to Canada were to be increased by some dislocation, for instance some

Another example of an advance in applied mathematical economics is presented by Professor Pigou's theory of joint production in railway rates. That theory has been severely criticised in the *Quarterly Review*. And it may be admitted that the criticism has weight so far as it is directed against the form of the exposition. That the gist of the theory should have been completely missed by a very discerning critic certainly argues some defect in the exposition. Perhaps it was injudicious on the part of Professor Pigou to use expressions which might suggest that his difference with the railway experts related only to definitions—"an accident of language." [1] Whereas the real issue relates not to the definition of terms, but to a distinction between things. There is a great difference between a condition of industry in which the cost of producing x of one commodity (say gas) is the same as that of producing x of that commodity plus y of some other commodity (say coke), and a condition in which the cost of producing x and y, say transportation of coal and transportation of copper, does, indeed, depend upon a single variable, but not now x or y, but z the *sum* of the two [2] (the number of tons of copper transported *plus* the number of tons of coal). The difference might be illustrated by the contrast between two methods which have been proposed for linking gold and silver so as to form a double standard of value. According to one method, which was called by its distinguished inventor "true bimetallism," [3] and has subsequently become known as "symmetallism," a sum of say £3 17*s*. 10½*d*. in standard money would procure a fixed fraction, say half, of an ounce of gold *plus* a fixed weight of silver, say $\frac{1}{2} \times 15\frac{1}{2}$ ounces (or, more generally, $\frac{1}{2} \times r$, r being a legalised ratio). According to the plan commonly known as bimetallism, the sum of £3 17*s*. 10½*d*. would procure *any* (proper) fraction of an ounce of gold, say $\frac{1}{t}$th of an ounce, *plus* a weight of silver equal to $15\frac{1}{2}\left(1 - \frac{1}{t}\right)$—or, more generally, $r\left(1 - \frac{1}{t}\right)$. Suppose

permanent impediment to its supply elsewhere, then it might be expected that—in the long run, and abstracting temporary disturbance—the offer on the part of Americans owning Niagara would be met by the demand for additional power on the part of the entrepreneurs in a large and flourishing Canada." *Scientia*, 1909, p. 90. (Above, c.)

[1] *Wealth and Welfare*, Part II. chap. xiii. § 3. *Cp.* § 4, where in denying a proposition about a certain kind of industry it is argued: "This is not Joint Supply."

[2] More exactly a *linear function* of the two quantities, say $ax + by$, where a and b are constants.

[3] Evidence of Professor Marshall before the Precious Metals Commission, 1887.

hasty thinkers to have confused the properties of these two very different systems. It would probably not be the best method of combating the confusion to ascribe it to an accident of language, a wrong definition of the term " bimetallism." Such a method of attack might seem to be met by the defence : " it would be strange if . . . experts, practical and theoretical, have fallen into a gross error by not understanding the words they use " (*Quarterly Review*, p. 421). Upon which we remark that the railway experts no doubt attached a clear conception to the words they used; but it was not an appropriate conception.[1] The subtlety of their thought was not equal to the subtlety of the distinctions existing in the nature of things.

For an examination of these delicate, but vital, distinctions the reader is referred to former articles in the ECONOMIC JOURNAL.[2] They are summarily re-stated here,[3] in the new light which has been thrown on the subject by recent publications. It is proper to begin with the simple case defined by Mill, " when the same outlay would have to be incurred for either of the two [products] if the other is not wanted or used at all." Professor Fanno expresses this datum by treating the ratio between the quantities of the two products as a constant, viz. K. He represents the real unity underlying the apparent duality of the products by an appropriate unit. Thus equipped he successfully attacks the main problems which the case presents; investigating the effects of a change in the demand of one or other of the commodities, of a tax or a bounty, now under the regime of competition, now under that of monopoly. The interest of these investigations extends beyond the simple case to which they primarily relate. The light which is applied at this particularly accessible point illuminates the comparatively inaccessible regions in the neighbourhood. This extension of illumination is effected by Professor Fanno through the introduction of a change in the value of the constant " K." This statement of the general problem has an advantage in respect of simplicity over that which has been given in a former number of the ECONOMIC JOURNAL.[4] The relation between the two presentations might be illustrated by the contrast between the modern and the older method of representing the relation between demand and price. This relation is

[1] Compare Whewell, *Inductive Sciences*, as to the part which clear and appropriate conceptions have played in scientific discovery.

[2] See XIII. p. 556, *et seq.* See Index, s.v. *Joint Production.*

[3] Before striking at what is here said the dissentient reader is requested to attend to what has been there said.

[4] IV. 54.

now commonly expressed by a curve or symbols. But the Ricardians preferred to say that "although the demand should be doubled, trebled, or quadrupled," [1] the price will ultimately fall to that "natural" price which is fixed by the cost of production. But the cost of production might vary; and in the case of agriculture at least it might vary in consequence of a change in the amount demanded.

As we leave the hard and fast limit formed by the classical instance of gas and coke we come to cases like wool and mutton, beef and hides, which are joint products in "a more partial sense," as Mill says; [2] a sense which has been made clearer by mathematical writers, in particular Dr. Marshall [3] and Mr. Flux.[4] "If we suppose the degrees of complementariness to be gradually diminished we shall pass through the zero point of absolute independence to a relation which may be distinguished as *rival* production; when the increased production of one commodity renders the increase of the other more difficult." [5] At the limit of this class, at the extreme which is opposite to the limit of Joint Production proper formed by Mill's instance, gas and coke, is placed the case now under consideration, where the cost of production depends on the simple sum of two (or more) quantities of product $(x + y)$, or, more generally, the *weighted* sum $(ax + by)$, where a and b are numerical constants, the dependence not being of the simplest sort, the cost not simply proportional to the said sum, but some *function* thereof, as may be expected where there are *general* expenses. The conception of this case as an extreme limit of rival production is countenanced by Mr. Johnson's parallel enunciation with respect to joint *demand*. He thus describes the "extreme cases" in which "the curves of utility degenerate into a series of parallel straight lines. "Here we may call x and z strictly or absolutely *competitive; i.e.*, any

[1] Ricardo, *Political Economy*, ch. xxx.

[2] *Political Economy*, III. ch. xvi. § 1.

[3] *Principles of Economics*, *sub voce* Joint Supply.

[4] *Economic Principles*.

[5] Quoted from the discussion of the subject in an early paper (V. 54, referring to VI.), in which context, it may be as well to caution the possible reader, there are some bad misprints, noticed among the errata in the decennial index. In a later paper (XIII. 558) it is proposed to use the term "disjunctive" as opposite to "Joint" (production). The class seems not to have been named by other writers, and perhaps with reason, since without further limitations it is too wide to be of service. It will apply to almost all economic production if in accordance with the first of the passages referred to it is exemplified wherever "a limited amount of time, strength or resources may be spent on either of two sorts of otherwise unconnected production." It might be better to limit the class to cases where there is a more active technical incompatibility between two kinds of production.

given amount of x gives the same utility as a proportional amount of z" (*loc. cit.*, p. 495). Substitute "y" for "z," "cost" for "utility," and for "curves of utility" "curves of equal cost," and you have a statement corresponding to ours.

Such, then, is the case to which it is proposed to attribute properties proper to Mill's case of gas and coke, identifying two limiting cases which are at the opposite extremes of opposed categories.[1] It is as if you were to attribute the same properties to the extreme cases of Diminishing Returns and its opposite; identifying the case in which no amount of additional outlay will produce any additional return and a virgin soil or nascent industry in which Increasing Returns operate with the greatest activity. Against this inaccuracy of language and thought Professor Fanno lends the weight of his authority. In a passage too long to be quoted here in full he thus characterises the "broad" sense of joint cost which has been opposed to Professor Pigou's narrow [*ristretto*] sense. "This excessive [*soverchia*] extension of the concept joint cost does not seem to us correct or scientifically rigorous. For every group of phenomena formed by any classification ought to comprise phenomena which resemble each other [*che sieno fra loro omogenei*] not merely superficially [*formalmente*] but in material respects [*sostanzialmente*]. Now this is far from being the case with the group denoted by 'Joint Supply' in the broad sense of the term. In the two cases [the broad and the limited sense] the character of the correlation between the prices is different; their behaviour [*comportamento*] and laws are different. We therefore reject the broad conception of joint cost as too vague and indeterminate." Professor Fanno writes with a knowledge of the vigorous but courteously conducted controversy —"*una vivace ma obiettiva polemica*"—between Professor Pigou and Professor Taussig in the *Quarterly Journal of Economics* (1913).

We are not so much concerned to prove that Professor Pigou's definition is the best, as to exhibit the importance of the new propositions which justify the definition. They relate chiefly to the characteristics of joint cost in the "broad" sense, what we have described as the limiting case of "competitive production." These properties are quite different from those which are com-

[1] Under the circumstances carefully defined by Professor Pigou, p. 218 (latter part). Compare ECONOMIC JOURNAL, Vol. XXI. p. 565; with reference to *certain magnitudes* of the increment Δx it might be impossible to increase the production of x by Δx without diminishing the cost of increasing y (the characteristic of Joint Production proper), *even though* the total cost was of the form $F(ax + by)$.

monly ascribed to joint cost of production. Of the latter, J. S. Mill writes: "Since cost of production here fails us, we must revert to a law of value anterior to cost of production, and more fundamental, the law of demand and supply." [1] So Professor Nicholson, referring to the case of joint products: "In this case the law of value is that the normal price of the two together is determined by the aggregate cost of production; and that the relative prices of the joint products, or the distribution of the aggregate between them, depends upon the demand and supply." [2] Well, the relative prices of the products, in the limiting case in question, do *not* depend upon the demand and supply in the sense which is evidently intended, the sense in which value depending on demand and supply only is opposed by J. S. Mill and the older writers to value depending on cost of production. It is *not* necessary now to "revert to a law anterior to cost of production." [3] It is as true now as in the normal case of value said to "depend on cost of production," that "although the demand should be doubled, trebled, or quadrupled," [4] prices tend to be the same if costs of production remain the same. Now, as in the normal case, "it is the cost of production which must ultimately regulate the price of commodities." [5] Only the regulating cost of production is not now, as usually to be understood, the total cost, but the prime cost, the cost of adding a unit of either product, other things being the same. Thus, suppose that the total cost of transporting x tons of coal *plus* y tons of copper depends only on $x + y$.[6] It follows from first principles that the increment in the total cost due to the increase (*ceteris paribus*) of x by any small weight τ is equal to τ multiplied by the rate at which the total cost increases with the increase of x. But that rate is the same as the rate at which the total cost increases with the increase of $x + y$, or of y only (x remaining constant).[7] Accordingly, the prime cost of transporting (a small unit of) coal will be the same as that of transporting copper. Therefore, according to the rule just now given, the price of transporting a ton of coal tends to be the same as the price of transporting a ton of copper, the general expenses being distributed equally between the two commodities. Again, supposing that the cost of transporting silver

[1] *Political Economy*, III. xvi. 1.

[2] *Principles of Political Economy*, Vol. II. p. 52. Cp. *Quarterly Review*, *loc. cit.*, p. 421.

[3] Any more than it is always, even in the general case of value said by Mill to depend on cost of production proper. [4] Ricardo as quoted above, p. 198.

[5] Ricardo, *loc. cit.* [6] *Cp.* XIII. p. 560.

[7] If the total cost of producing $x + y$ is $F(x + y)$ the rate at which this increases with the increase of either x or y is $F'(x + y)$.

and gold depends only on the weight (concrete circumstances, such as differences of risk and insurance, being abstracted); then in a regime of bimetallism (in the ordinary sense of that term) the prime cost of transporting a thousand pounds sterling in silver would be $15\frac{1}{2}$ times the prime cost of transporting a thousand pounds sterling in gold (supposing the bimetallic ratio to be $15\frac{1}{2} : 1$); and, accordingly, the price of transporting a thousand pounds sterling in silver would tend to be $15\frac{1}{2}$ times the price of transporting that sum in gold.

It may be objected that this proposition is not confirmed by observation; the predicated exact correspondence between prime cost and price is not observable throughout the real world of industry. It may be replied that the requisite condition of perfectly competitive production is often not perfectly fulfilled; and, further, that even when the condition is at least approximately fulfilled, the resulting tendency is obstructed by an element of monopoly. We may have to rely largely on general reasoning, of a piece with that which is generally accepted in proof of a correspondence between value and cost of production in ordinary cases. In the present case, indeed, the importance of the tendency consists in its not being fulfilled in practice. There is afforded a rule for the regulation of industries in which the fulfilment of the tendency is obstructed by monopolistic friction. This regulative idea may well be of far-reaching importance. A check is thus given to the spread of the heresy, as it would have seemed to the older economists, that there is no presumption against charging different prices for like services: electric power may properly be sold at different prices according to the use for which it is destined, a doctor may fairly vary his fees according to the means of the patient, and so on.[1]

It must be remembered that the tendency which has been stated rests on the sort of probability which is here all along understood. The rule holds good, *primâ facie* and provisionally, until cause to the contrary be shown. One cause that is to be looked for is a condition of supply and demand such that without discrimination contrary to the rule production cannot be made to pay. With reference to industries generally, "in an industry selected at random," as Professor Pigou has it,[2] it appears improbable, for reasons given by him, that this sort of exception should occur.

It is a question of great practical interest whether the excep-

[1] *Cp.* Acworth, *Railway Economics*, ch. ix.

[2] *Wealth and Welfare*, Part II. ch. xii., especially §§ 15, 16, 17.

tion is likely to occur in the class of industries for the regulation of which a rule is much required, " public works," in M. Colson's phrase, and, in particular, railways. Railway experts in general bear witness in favour of a discrimination inconsistent with the rule. But their testimony is obscured by the confusion attending the double signification of " joint cost." The answers of experts are then most authoritative when they are addressed to questions framed by correct theory. Pending further discussion, we are disposed to agree with Professor Pigou that, while discrimination or the " value of service principle " is required at a certain early stage of a country's development, the " cost of service principle " should be the rule for more developed countries.[1] Thus considerable weight is added to the reasons in favour of " the cost of service principle," which is now being enforced by the Interstate Commerce Commission.

At the same time, the weight on the other side of the balance is lightened. For the Competition to which advocates of discrimination appeal—Professor Hugo Meyer notably [2]—is found to be of the nature of *Duopoly*, a species of competition which has not in its favour the same presumptions as that which Professor Pigou calls " simple competition." [3]

Many other examples of mathematical reasoning based upon Probabilities and bearing upon practice are to be found in Professor Pigou's work. We do not conceal that there is something tentative in these applications. Nor do we put forward our explanation of their philosophical basis as final. Rather, in the words of the philosopher who first divined the deep connection between Probabilities in a technical sense and Induction in general,[4] we " shall think it sufficient if the present hints excite the curiosity of philosophers and make them sensible how defective all common theories are in treating of such curious and sublime subjects."

[1] *Wealth and Welfare, loc. cit.*, § 10. Compare as to objections and qualification, XIV.

[2] *Governmental Regulation of Railway Rates, passim*, and especially with reference to the system of " blanketing," which the Interstate Commerce Commission has since limited.

[3] *Wealth and Welfare*, Part II. ch. viii. and ch. vii. §§ 12–14; ch. xvi. §§ 2–3.

[4] Hume, " Of Probability," sec. vi. of *An Enquiry Concerning Human Understanding* (*Essays*).

END OF VOL. II.